

Twisted virtual biracks and their twisted virtual link invariants

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Abstract

A virtual link can be understood as a link in a trivial I -bundle over an orientable compact surface with genus. A *twisted virtual link* is a link in a trivial I -bundle over a not-necessarily orientable compact surface. A *twisted virtual birack* is an algebraic structure with axioms derived from the twisted virtual Reidemeister moves. We extend a method previously used with racks and biracks to the twisted case to define computable invariants of twisted virtual links using finite twisted virtual biracks with birack rank $N \geq 1$. As an application, we classify twist structures on the virtual Hopf link.

KEYWORDS: Virtual links, Twisted virtual links, biracks, counting invariants

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1 Introduction

In [10, 8] *virtual knots and links* and equivalently *abstract knots links* respectively were introduced. Virtual knots were initially conceived as combinatorial objects, Reidemeister equivalence classes of Gauss codes, where abstract knots were geometric in nature, knot diagrams drawn on minimalistic supporting surfaces. In [3] a geometric interpretation of virtual and abstract links as isotopy classes of simple closed curves in I -bundles over compact oriented surfaces modulo stabilization moves was developed.

In recent work such as [1, 7], virtual and abstract links are extended to allow compact non-orientable supporting surfaces; the resulting links are called *twisted virtual links*. Invariants of twisted virtual links such as the twisted Jones polynomial and the twisted knot group have been introduced and studied. Twisted virtual knot theory is very new, with many interesting open questions, such as finding a supporting surface of minimal genus for a given twisted virtual knot or link.

Quandle- and biquandle-based invariants of twisted virtual links were first considered in [9]. In [14] a counting invariant of unframed oriented classical and virtual knots and links was defined using labelings by finite racks and extended to finite biracks in [15]. In this paper we extend the birack counting invariant to the case of twisted virtual links.

The paper is organized as follows: in section 2 we recall the basics of twisted virtual link theory from [1, 7] and make a few observations which will be useful in later sections. In section 3 we recall twisted virtual biracks give some examples. In section 4 we define the twisted virtual birack counting invariant and provide examples and sample computations of the invariant. As an application, we classify twist structures on the virtual Hopf link. We end with a few open questions for future research in section 5.

2 Twisted Virtual Links

Twisted virtual links were introduced in [1] and subsequently studied in works such as [7, 9]. Twisted virtual links extend the concept of *virtual links* from previous work [10, 8]; where virtual links arise by drawing link diagrams on compact orientable surfaces with nonzero genus, twisted virtual links arise when we draw link diagrams on compact surfaces allowing nonzero genus and nonzero cross-cap number.

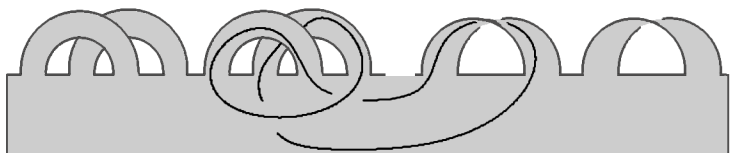
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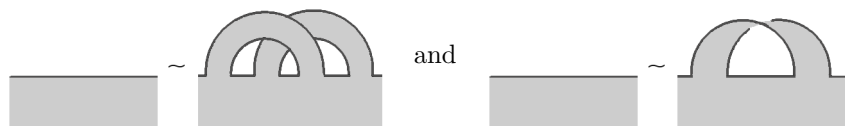
Knot and link diagrams are usually drawn on flat paper without explicitly specifying a supporting surface Σ on which the knot diagram is drawn. If we do explicitly draw Σ , we have a *link-surface* diagram. Often we will remove a disk from $\Sigma \setminus L$ so we can flatten Σ as depicted. Virtual crossings correspond to crossed bands while classical crossings correspond to crossings drawn on Σ .



Geometrically, a twisted virtual link is a stable equivalence class of simple closed curves in an I -bundle, i.e. an ambient space obtained by thickening the surface Σ on which the link diagram is drawn. Here “stable equivalence” means that in addition to ambient isotopy of the link within the thickened surface, we can *stabilize* the surface Σ by adding or deleting torus summands or cross caps not containing the link. If we remove a disk from $\Sigma \setminus L$ and flatten the resulting Σ' in the usual way to get



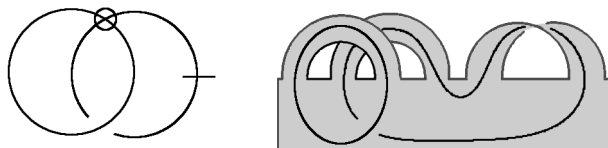
then stabilization moves have the form



We can represent twisted virtual links combinatorially without having to draw the supporting surface Σ by representing crossings arising from genus in Σ with circled self-intersections known as *virtual crossings* (see [10, 8]) and representing places where our link traverses a cross cap in Σ with a small bar.

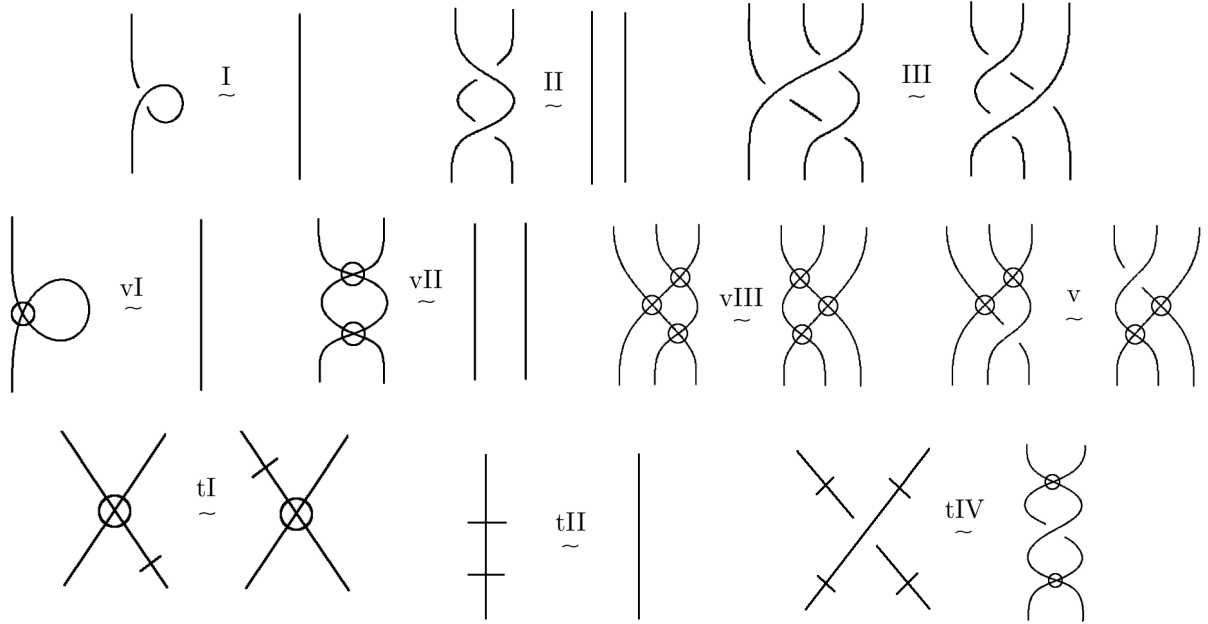


Then for instance the twisted virtual Hopf link diagram below corresponds to the link-surface diagram shown.

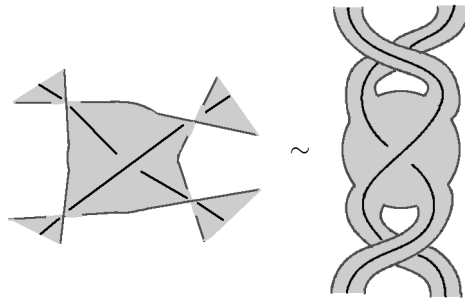


The portions of a twisted virtual link diagram L between overcrossings, undercrossings, virtual crossings and bars are *semiarcs*. For instance, the twisted virtual Hopf link diagram above has six semiarcs.

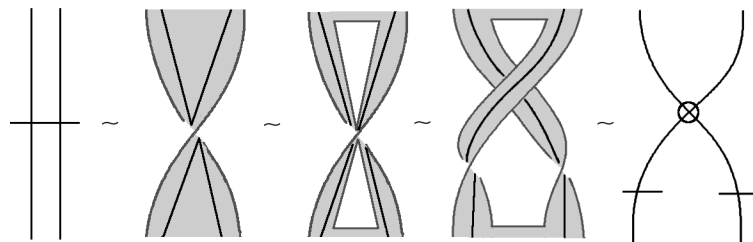
In [1] it is shown that stable isotopy of twisted virtual links corresponds to the equivalence relation on twisted virtual link diagrams generated by the *twisted virtual Reidemeister moves*:



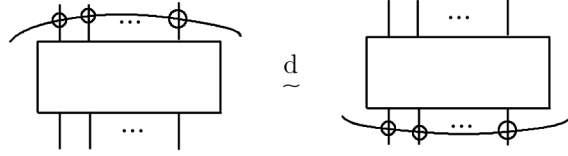
Each of these moves can be understood in terms of link-surface diagrams or abstract link diagrams; for instance, the last move looks like:



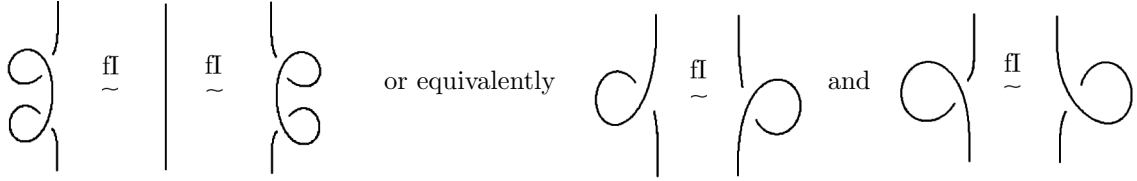
Note that we do not need twist bars covering multiple strands since for any two neighboring strands going through a twisted band, we can remove a disc from the surface between the strands and replace the multi-strand bar with two bars and a virtual crossing:



The four virtual moves together imply the *detour move*, which says that a strand with only virtual crossings can be moved past any tangle containing classical or virtual crossings; move tI implies that strands with only virtual crossings can detour past twist bars as well.



Replacing the usual classical Reidemeister type I move with the blackboard framed type I moves



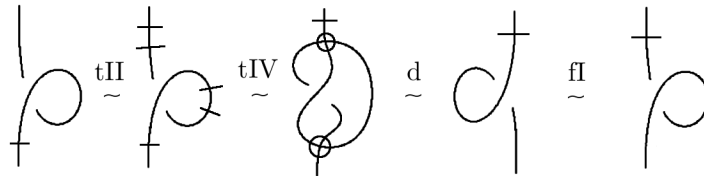
yields *blackboard framed twisted virtual isotopy*. Including orientations on the link components gives *oriented blackboard framed twisted virtual isotopy*.

We will primarily be interested in using invariants of oriented blackboard framed twisted virtual isotopy to define an invariant of oriented unframed twisted virtual isotopy analogous to those defined in [14] and [15].

We will find the following observations useful in the next section.

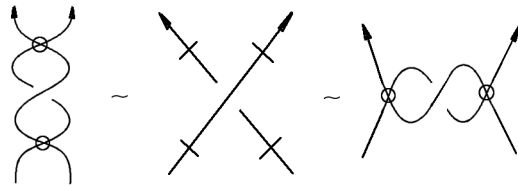
Lemma 1 *A twist bar can be moved past a classical kink with virtual twisted blackboard framed isotopy moves.*

Proof.

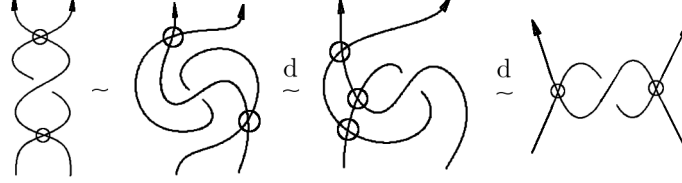


□

Lemma 2 *The two oriented versions of the last twisted virtual move are equivalent, i.e we have*



Proof.



□

3 Twisted Virtual Biracks

We begin with a definition slightly modified from [9].

Definition 1 Let X be a set and $\Delta : X \rightarrow X \times X$ the diagonal map $\Delta(x) = (x, x)$. A *twisted virtual birack* is a set X with invertible maps $B, V : X \times X \rightarrow X \times X$, and an involution $T : X \rightarrow X$ satisfying the axioms below where F_j denotes a map F followed by projection onto the j th component:

- (i) B and V are *sideways invertible*: there exist unique invertible maps $S : X \times X \rightarrow X \times X$ and $vS : X \times X \rightarrow X \times X$ such that for all $x, y \in X$ we have

$$S(B_1(x, y), x) = (B_2(x, y), y) \quad \text{and} \quad vS(V_1(x, y), x) = (V_2(x, y), y);$$

- (ii) The compositions $(S^{\pm 1} \circ \Delta)_k$ and $(vS^{\pm 1} \circ \Delta)_k$ are bijections for $k = 1, 2$;

- (iii) $(vS \circ \Delta)_1 = (vS \circ \Delta)_2$;

- (iv) B and V satisfy the set-theoretic Yang-Baxter equations:

$$(B \times \text{Id}_X)(\text{Id}_X \times B)(B \times \text{Id}_X) = (\text{Id}_X \times B)(B \times \text{Id}_X)(\text{Id}_X \times B),$$

$$(V \times \text{Id}_X)(\text{Id}_X \times V)(V \times \text{Id}_X) = (\text{Id}_X \times V)(V \times \text{Id}_X)(\text{Id}_X \times V),$$

and

$$(B \times \text{Id}_X)(\text{Id}_X \times V)(V \times \text{Id}_X) = (\text{Id}_X \times V)(V \times \text{Id}_X)(\text{Id}_X \times B);$$

- (v)

$$(T \times \text{Id})V = V(\text{Id} \times T) \quad \text{and} \quad (\text{Id} \times T)V = V(T \times \text{Id}),$$

- (vi)

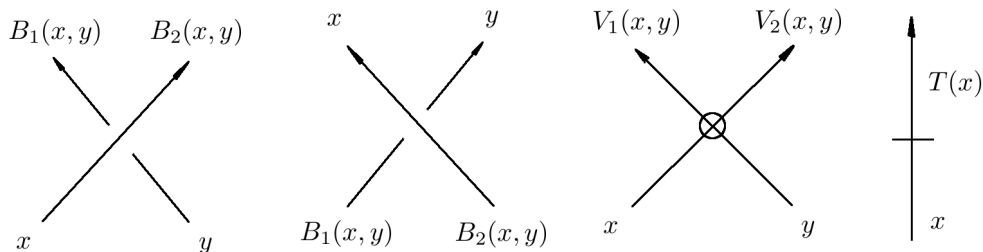
$$(T \times T)B(T \times T) = VBV.$$

If we also have

- (iii') $(S \circ \Delta)_1 = (S \circ \Delta)_2$,

then X is a *twisted virtual biquandle*.

The twisted virtual birack axioms are obtained from the blackboard framed twisted virtual Reidemeister moves using the following semiarc-labeling scheme:



See [9] for more details.

Let X and Y be twisted virtual biracks with maps B_X, V_X, T_X and B_Y, V_Y, T_Y respectively. As with other algebraic structures, we have the following common notions:

- A map $f : X \rightarrow Y$ is a *homomorphism of twisted virtual biracks* if

$$B_Y(f \times f) = (f \times f)B_X, \quad V_Y(f \times f) = (f \times f)V_X \quad \text{and} \quad T_Y f = fT_X;$$

and

- If $Y \subset X$, then Y is a *twisted virtual subbirack* of X provided

$$B_Y = B_X I, \quad T_Y = T_X I, \quad \text{and} \quad T_Y = T_X I$$

where $I : Y \rightarrow X$ is inclusion.

The map $\pi = (S^{-1} \circ \Delta)_1 \circ (S^{-1} \circ \Delta)_2^{-1}$ represents going through a positive kink; this is known as the *kink map*. A birack is *biquandle* if its kink map is the identity; otherwise, π is an element of S_n and its order N is the *birack rank* or *birack characteristic* of X .

Remark 1 If X is a twisted virtual birack, then the map B defines a birack structure on X and the map V defines a semiquandle structure on X , i.e. a biquandle structure with $V^2 = \text{Id}$. The pair B, V then defines a *virtual birack* structure on X , i.e. a birack structure with a compatible semiquandle structure. Thus, a twisted virtual birack is a virtual birack with a compatible twist map T . See [6, 12].

Moreover, if the birack rank of (X, B) is 1, then we have a *twisted virtual biquandle*. If $B_2(x, y) = x$ for all $x \in X$, then we have a *twisted virtual rack*, and a twisted virtual rack which is also a twisted virtual biquandle is a *twisted virtual quandle*. We summarize the different structures with a table:

	$N = 1$	$N \neq 1$	
$B_2 = \text{Id}_X$	quandle	rack	• If $B^2 = \text{Id}_{X \times X}$, replace “bi-” with “semi-”
$B_2 \neq \text{Id}_X$	biquandle	birack	• If $V(x, y) \neq (y, x)$, add “virtual”
			• If $T \neq \text{Id}_X$, add “twisted”

Example 1 Let X be a commutative ring with multiplicative group of units X^* , $t, r \in X^*$ and $s \in X$ satisfying $s^2 = (1 - tr)s$; then X is a birack with $B(x, y) = (ty + sx, rx)$ known as a (t, s, r) -birack (see [14]). If we likewise choose $v, w \in X^*$, $u \in X$ satisfying $u^2 = (1 - vw)u$ and set $V(x, y) = (vy + ux, wx)$, then $V^2 = \text{Id}$ requires that

$$x = (vw + u^2)x + uv y \quad \text{and} \quad y = vwy + wux$$

which in turn implies $w = v^{-1}$ and $u = 0$. Note that $u^2 = 0^2 = (1 - 1)0 = (1 - vw)u$ and the condition $u^2 = (1 - vw)u$ is automatic in this case. Thus, our virtual operation becomes $V(x, y) = (vy, v^{-1}x)$. The mixed virtual move then requires that multiplication by v commutes with multiplication by t, s and r .

Choosing $T \in X^*$ so that $T : X \rightarrow X$ is given by $T(x) = Tx$, the twisted moves then require that $T^2 = 1$, multiplication by v commutes with multiplication by T , and that

$$T^2 rx = rx = v^{-2}tx + sy, \quad \text{and} \quad v^2 ry = T^2 ty + T^2 sx = ty + sx.$$

Comparing coefficients, we see that we need $s = 0$ and $v^2 r = t$. Since $s = 0$ implies $s^2 = (1 - tr)s$, the conditions on our coefficients reduce to $t, r, v, T \in X^*$ with $T^2 = 1$ and $v^2 r = t$. Thus, for any commutative ring X with units t, r, v, T satisfying $v^2 r = t$ and $T^2 = 1$, we have a twisted virtual birack structure on X defined by

$$B(x, y) = (ty, rx), \quad V(x, y) = (vy, v^{-1}x), \quad \text{and} \quad T(x) = Tx.$$

Remark 2 It is curious that unlike the case of virtual biracks, simply making the twist operation trivial does not give a natural embedding of the category of virtual biracks into the category of twisted virtual biracks. More precisely, every birack is a virtual birack with virtual operation $V(x, y) = (y, x)$; on the other hand, if we have trivial operations at the virtual crossings and twist bars, then the twisted move tv requires the component maps of B to be equal, i.e. that $y_x = y^x$, a condition which is false for most biracks. In particular, for a given virtual birack, the set of compatible twist structures may be empty.

Example 2 We can represent a twisted virtual birack structure on a finite set $X = \{x_1, \dots, x_n\}$ with an $n \times (4n + 1)$ -matrix $M_X = [U|L|vU|vL|T]$ with $n \times n$ blocks U, L, vU, vL encoding the operations B, V and an $n \times 1$ block T encoding the involution T in the following way:

- if $B(x_i, x_j) = (x_k, x_l)$ we set $U_{ji} = k$ and $L_{ij} = l$ (note the reversed order of the subscripts in U ; this is for compatibility with previous work);
- if $V(x_i, x_j) = (x_k, x_l)$ we set $vU_{ji} = k$ and $vL_{ij} = l$, and
- if $T(x_i) = x_j$ we set $T[i, 1] = j$.

Every finite twisted virtual birack can be encoded by such a matrix, and conversely an $n \times (4n + 1)$ matrix with entries in $\{1, 2, \dots, n\}$ defines a twisted virtual birack provided the twisted virtual birack axioms are all satisfied by the maps B, V and T defined by the matrix.

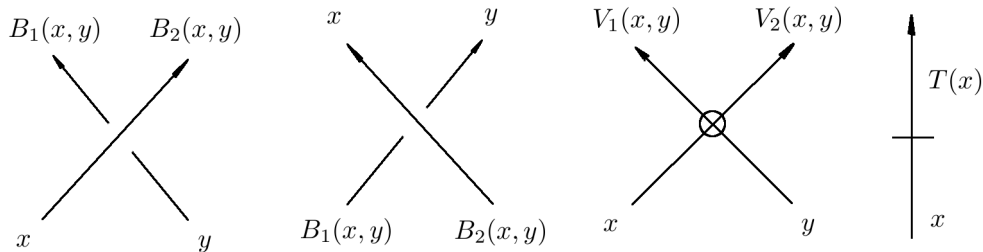
For instance, our Python computations reveal that there are eight twisted virtual birack structures on the set $X = \{x_1, x_2\}$, given by the matrices

$$\begin{aligned} & \left[\begin{array}{cc|cc|cc|cc|c} 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 \end{array} \right], \left[\begin{array}{cc|cc|cc|cc|c} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right], \\ & \left[\begin{array}{cc|cc|cc|cc|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \end{array} \right], \left[\begin{array}{cc|cc|cc|cc|c} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \end{array} \right], \\ & \left[\begin{array}{cc|cc|cc|cc|c} 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \end{array} \right], \left[\begin{array}{cc|cc|cc|cc|c} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{array} \right], \\ & \left[\begin{array}{cc|cc|cc|cc|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{array} \right], \left[\begin{array}{cc|cc|cc|cc|c} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 2 \end{array} \right]. \end{aligned}$$

4 Counting Invariants

Let L be a blackboard framed twisted virtual link diagram. Recall that a semiarc is a portion of L between classical or virtual crossing points or twist bars. Let us define a *classical semiarc* as a portion of a twisted virtual knot or link obtained by dividing at twist bars and classical over and under crossing points only; classical semiarcs may contain virtual crossing points.

Definition 2 Let L be a twisted virtual link diagram and X a twisted virtual birack. A *twisted virtual birack labeling* of L by X , or just an *X -labeling* of L , is an assignment of an element of X to every semiarc in L such that at every classical crossing, virtual crossing, and twist bar we have



The twisted virtual birack axioms are consequences of the oriented blackboard framed twisted virtual Reidemeister moves using the labeling conventions in definition 2. Thus, by construction we have

Theorem 3 If L and L' are twisted virtually blackboard framed isotopic twisted virtual links and X is a finite twisted virtual birack, then the number of X -labelings of L equals the number of X -labelings of L' .

As with quandle labelings of oriented classical links, rack labelings of blackboard framed classical links, etc., an X -labeling of a twisted virtual link diagram L can be understood as a homomorphism $f : TVB(L) \rightarrow X$ of twisted virtual biracks where $TVB(L)$ is the *fundamental twisted virtual birack* of L defined below. More precisely, let G be a set of symbols, one for each semiarc in L , and define the set of *twisted virtual birack words in G* , $W(G)$, recursively by the rules

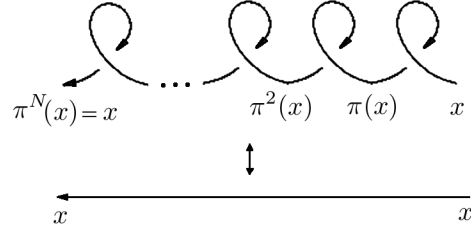
- $g \in G \Rightarrow g \in W(G)$ and
- $g, h \in W(G) \Rightarrow B_k^{\pm 1}(g, h), S_k^{\pm 1}(g, h), V_k^{\pm 1}(g, h), vS_k^{\pm 1}(g, h), T(g) \in W(G)$ for $k = 1, 2$.

Then the *free twisted virtual birack* on G is the set of equivalence classes in $W(G)$ modulo the equivalence relation generated by the twisted virtual birack axioms, and the *fundamental twisted virtual birack of L* is the set of equivalence classes of elements of the free twisted virtual birack on G modulo the equivalence relation generated by the crossing relations. Both sets are twisted virtual biracks under the operations

$$B([x], [y]) = ([B_1(x, y)], [B_2(x, y)]), \quad V([x], [y]) = ([V_1(x, y)], [V_2(x, y)]), \quad T([x]) = [T(x)]$$

where $[x]$ is the equivalence class of $x \in W(G)$.

In [15] it is observed that the number of labelings by a rank N birack of a link diagram is unchanged by the N -phone cord move:



In particular, the number of labelings is periodic in the writhe of each component of L with period N , and consequentially the sum of the numbers of labelings over a complete period of writhes mod N forms an invariant of unframed isotopy. We would like to extend this invariant to the category of twisted virtual biracks.

Definition 3 Let L be a twisted virtual link with c components and X a twisted virtual birack with rank N . The *integral twisted virtual birack counting invariant* is the number of X -labelings of L over a complete period of blackboard framings of L mod N . That is,

$$\Phi_X^{\mathbb{Z}}(L) = \sum_{\mathbf{w} \in (\mathbb{Z}_N)^c} |\text{Hom}(TVB(L, \mathbf{w}), X)|$$

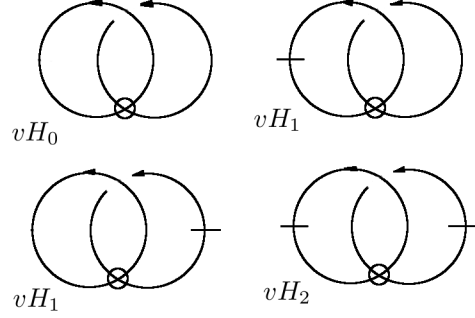
where $TVB(L, \mathbf{w})$ is a diagram of L with framing vector $\mathbf{w} \in (\mathbb{Z}_N)^c$.

By construction, we have

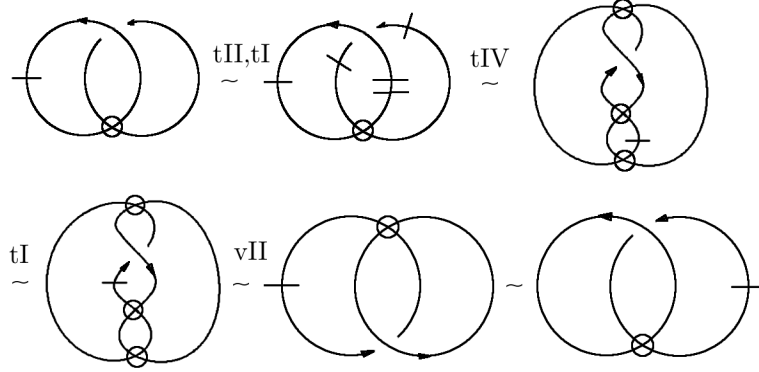
Theorem 4 If L and L' are twisted virtually isotopic twisted virtual links and X is a finite twisted virtual birack, then $\Phi_X^{\mathbb{Z}}(L) = \Phi_X^{\mathbb{Z}}(L')$.

Starting with a virtual link diagram, it natural to ask which placements of twist bars yield distinct twisted virtual links. In light of moves tI and tII, we can place at most one twist bar on any portion of the knot between classical crossing points, i.e. on any classical semiarc.

Example 3 The *virtual Hopf link* is the smallest (in terms of classical and virtual crossing numbers) nontrivial virtual link with two components. It has only two classical semiarcs, so there are $2^2 = 4$ potentially different twisted links which project to the virtual Hopf link under removal of twist bars.



We can see that two of the possibilities are equivalent using twisted virtual isotopy moves:



We note that in this move sequence, the twist bar appears to move from one component of the link to the other; however, this is an illusion due to the symmetry of the link; in fact, the twisted virtual moves have changed the over/under relationship of the crossing.

On the other hand, we can use the integral counting invariant $\Phi_X^{\mathbb{Z}}(L)$ with respect to various twisted virtual biracks to show that the links with two twist bars, one twist bar and zero twist bars are not equivalent. Let X_1 be the twisted virtual birack with matrix

$$M_{X_1} = \left[\begin{array}{cc|cc|cc|cc|cc} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right]$$

Then we have $\Phi_{X_1}^{\mathbb{Z}}(vH_0) = 4$, $\Phi_{X_1}^{\mathbb{Z}}(vH_1) = 0$, and $\Phi_{X_1}^{\mathbb{Z}}(vH_2) = 0$. Similarly, let X_2 be the twisted virtual birack with matrix

$$M_{X_2} = \left[\begin{array}{ccc|ccc|ccc|ccc|c} 2 & 2 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \end{array} \right]$$

Then we have $\Phi_{X_2}^{\mathbb{Z}}(vH_0) = 5$, $\Phi_{X_2}^{\mathbb{Z}}(vH_1) = 3$, and $\Phi_{X_2}^{\mathbb{Z}}(vH_2) = 5$.

Finally, we note that as in the case of biracks, we have several enhancements of the twisted virtual birack counting invariant. An *enhancement* associates an X -labeled move invariant signature to each labeling of a twisted virtual link diagram so that instead of simply counting labelings, we collect the signatures to get a multiset whose cardinality recovers the counting invariant but is in general a stronger invariant. We usually convert the multiset into a polynomial for ease of comparison by taking a generating function. The standard enhancements include the following:

- *Image enhancement.* Given a valid labeling $f : TVB(L) \rightarrow X$ of the semiarcs in twisted virtual link diagram L of c components by a twisted virtual birack X of rank N , the image of f is an invariant of twisted virtual isotopy. From a labeled link diagram, we can compute $\text{Im}(f)$ by taking the closure under the operations $B_1(x, y), B_2(x, y), V_1(x, y), V_2(x, y)$ and $T(x)$ of the set of all elements of Y appearing as semiarc labels. Then we have an enhanced invariant

$$\Phi_X^{\text{Im}}(L) = \sum_{\mathbf{w} \in (\mathbb{Z}_N)^c} \left(\sum_{f \in \text{Hom}(TVB(L, \mathbf{w}), X)} u^{|\text{Im}(f)|} \right).$$

- *Writhe enhancement.* For this one, we simply keep track of which writhe vectors contribute which labelings. For a writhe vector $\mathbf{w} = (w_1, \dots, w_c)$, let us denote $q^{\mathbf{w}} = q_1^{w_1} \dots q_c^{w_c}$. Then the *writhe enhanced invariant* is

$$\Phi_X^W(L) = \sum_{\mathbf{w} \in (\mathbb{Z}_n)^c} |\text{Hom}(TVB(L, \mathbf{w}), X)| q^{\mathbf{w}}.$$

- *Twisted virtual birack polynomials.* Let X be a finite twisted virtual birack with birack matrix $[M_1|M_2|M_3|M_4|M_5]$. For each element $x_k \in X = \{x_1, \dots, x_n\}$, let

$$c_i(x_k) = |\{j : M_i[x_j, x_k] = x_j\}| \quad \text{and} \quad r_i(x_k) = |\{j : M_i[x_k, x_j] = x_k\}|.$$

Then for any twisted virtual subbirack $Y \subset X$, the sub-TVG polynomial of Y is

$$p_{Y \subset X} = \sum_{x \in Y} \left(\sum_{i=1}^5 t_i^{c_i(x)} s_i^{r_i(x)} \right).$$

Then for each X -labeling f of L , the twisted virtual subbirack polynomial of the image of f gives an invariant signature, so we have the *twisted virtual birack polynomial enhanced invariant*

$$\Phi_X^p(L) = \sum_{\mathbf{w} \in (\mathbb{Z}_N)^c} \left(\sum_{f \in \text{Hom}(TVB(L, \mathbf{w}), X)} u^{p_{\text{Im}(f) \subset X}} \right).$$

See [13] for more.

5 Questions

In remark 2 we observed that trivial virtual and twisted operations do not generally give a birack the structure of a twisted virtual birack. In [9], a construction called the *twisted product* is given in which a birack B and a choice of automorphism of B are used to define a twisted virtual birack structure on the Cartesian product $B \times B$. What other ways are there to define a twisted virtual birack given a birack?

As with virtual knots, twisted virtual knot theory opens the possibility of new invariants of classical knots and link defined in terms of twisted virtual links, since classical links form a subset of twisted virtual links and any invariant of twisted virtual isotopy is *a fortiori* an invariant of classical links. What new invariants of classical and virtual links can only be defined using twisted virtual biracks?

We have identified a few enhancements of twisted virtual biracks; what are some additional enhancements? What is the best way to generalize virtual biquandle homology (see [2]) to define twisted virtual birack homology?

For classical knots, the fundamental quandle is a complete invariant up to ambient homeomorphism; it is conjectured (see [4]) that the fundamental biquandle is a complete invariant of virtual knots up to vertical mirror image. Is the fundamental twisted virtual biquandle a complete invariant of twisted virtual knots up to vertical mirror image? What conditions on twisted virtual knots with isomorphic fundamental twisted virtual birack suffice to guarantee that the knots are twisted-virtually isotopic?

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